利用有限積分轉換探討鋼筋混凝土之非線性行爲

Study on Non-linear Deflection of Reinforced Concrete Using Finite Integration Transform

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摘　　要

本文在探討鋼筋混凝土梁在bending force作用時之力學行爲。研究範圍由彈性領域至極限狀態。有關鋼筋混凝土之性質方面我們採用了Duffing model來描述其應力—應變關係間之非線性行爲。本研究證實了Duffing型非線性應力—應變關係式之適用性，同時也顯示當鋼筋混凝土簡支梁受到bending force作用時，根據混凝土之非線性關係可算出任意點之曲率，再利用有限積分轉換法可推出其deflection之理論式。由此理論式可算出鋼筋混凝土簡支梁由彈性狀態至破壞狀態之deflection。

關鍵詞：非線性行爲，鋼筋混凝土，變位，有限積分轉換。

ABSTRACT

This study aimed at an analysis of the mechanical behavior of reinforced concrete beams under bending forces, ranging from the elastic domain to the ultimate critical state. As for the mechanical property of the concrete, we adopted the Duffing model, in which the stress-strain relationship is non-linear. The results of our study prove that the use of a Duffing non-linear stress-strain relationship is appropriate. The study also demonstrated that deflection of a reinforced concrete simple beam between the elastic and ultimate states can be obtained by calculating the curvature of any given point based on its non-liner characteristics and obtaining a deflection equation from the curvature the finite integration transform.

Key word: non-linear, reinear, reinforced concrete, deflection, finite integration transform

1. INTRODUCTION

The ultimte state design method has gradually become a major approach to design civil engineering structures. This method takes the approach most suitable for each of the various ultimate states which may arise as a result of the loading applied to the structure. Generally, a study of a structural member's profile is carried out with the focus on the ultimate critical state of its rupture profile. It is, therefore, necessary to comprehend not only the strength of the member, but also its non-linear behavior and its mechanical behavior beyond the elastic domain up to the ultimate rupture state.

The aim of this study is to clarify the mechanical behavior of a reinforced concrete member under bending force starting from elastic to ultimate critical states. In this study, Duffing model is adopted as the non-linear stress-strain relation to clarify mechanical behavior. The study also present an analysis for the deflection of a reinforced concrete simple beam from elastic to ultimate states. The procedure is to obtain by calculating the curvature of any given section based on the above non-linear characteristics an and to obtain the deflection from the curvature through the finite integration transforms.

2. PREVIOUS STUDIES

A number of equations have so far been used in the study of concrete stress-strain curves. Examples are seen in the experimental equations by Saka[1)] and Fujita[2)] in which stress σ_C is represented by a cubic expression of the strain, as shown below. (ε_{CB}: strain at time of maximum compressive stress) Equation by Saka (concrete aged 28 days):

$$\sigma_C = A_1 \varepsilon_C + B_1 \varepsilon_C^2 + C_1 \varepsilon_C^3 \qquad (1)$$

where, $A_1 = E_{C1} = (2.367 + 0.0038\sigma_{CB}) \times 10^5$,

$B_1 = (-3.178 + 0.008\sigma_{CB}) \times 10^8$,

$C_1 = (10.67 - 0.0406\sigma_{CB}) \times 10^{10}$

Equation by Fujita:

$$\sigma_C = E_{C2}\varepsilon_C\left\{1 - (2 - 3\xi)\frac{\varepsilon_C}{\varepsilon_{CB}} + (1 - 2\xi)\left(\frac{\varepsilon_C}{\varepsilon_{CB}}\right)^2\right\} \qquad (2)$$

where, $\xi = \dfrac{\sigma_{CB}}{E_{C2}\varepsilon_{CB}}$, elastic coefficient

$$E_{C2} = 3.1 \times 10^3 \sigma_{CB}^{0.4},$$

σ_{CB}: maximum compressive stress

Given below is a general equation for the neutral axis ratio k and moment M_{Cr} *used in* strength calculations at the critical cracking state when a reinforced concrete member of sectional porfile ($b \times h$) is subjected to a bending force[2)].

$$k = \frac{B_2}{A_2}\left\{-1 + \left(1 + \frac{A_2 C_2}{B_2^2}\right)^{0.5}\right\} \qquad (3)$$

where, $A_2 = \alpha' - \dfrac{\alpha_T \sigma_{CT}}{E_{C2}\varepsilon_{CT}}$

$$B_2 = \frac{\alpha_T \sigma_{CT}}{E_{C2}\varepsilon_{CT}} + 0.5n(P_T + P_C)$$

$$C_2 = \frac{\sigma_T \sigma_{CT}}{E_{C2}\varepsilon_{CT}} + nP_T(1 - \mu) + nP_C\mu'$$

Ratio of tendsile reinforcement: $P_T = A_{ST}/bh$

Ratio of compression reinforcement:

$P_C = A_{SC}/bh$

Ratio of modulus of elasticity: $n = E_C/E_S$

E_C: elastic modulus of concrete

E_S: elastic modulus of reinforcement

A_{ST}: tensile reinforcement content

A_{SC}: compressive reinforcement content

Compression-side cover: $e' = \mu' h$

Tension-side cover: $e = \mu h$

Maximum tensile stress:

$\sigma_{CT} = 24 + 0.052\,(\sigma_{CB} - 200)$

Strain at maximum tensile stress: $\varepsilon_{CT} = (18.5 + 0.02\sigma_{CT}) \times 10^{-5}$ (σ_{CT} is in kg/cm^2)

$$\frac{M_{Cr}}{bh^2\sigma_{CT}} = \frac{\alpha'\beta' E_{C2}\varepsilon_{CT}}{\sigma_{CT}} \times \frac{k^3}{1-k} + \frac{nP_C E_{C2}\varepsilon_{CT}}{\sigma_{CT}} \times \frac{(k-\mu')^2}{1-k} + \alpha_T\beta_T(1-k)^2 + \frac{nP_T E_{C2}\varepsilon_{CT}}{\sigma_{CT}} \times \frac{(1-k-\mu)^3}{1-k} \quad (4)$$

where a $\alpha' = 0.5$, $\beta' = 0.593$, $\alpha_T = 0.783$, $\beta_T = 2/3$

The neutral axis ratio k and moment M_{ur} used in strength calculations at the member's ultimate state are expressed by the following general equation[3]:

In the single reinforcement case, the calculation is performed by adopting the following two equations considering the relationship between reinforcement yield stress σ_{SY} and the stress σ_{ST} acting on the tensional reinforcement.

Case I: $\sigma_{ST} = \sigma_{SY}$

$$k = \frac{P_T\sigma_{ST}}{\alpha\sigma_{CB}} \quad (5)$$

$$M_{ur} = \alpha k b d_1^2 \sigma_{CB}(1-\beta k) \quad (6)$$

Case II: $\sigma_{ST} < \sigma_{SY}$

$$k = -\frac{P_T E_S \varepsilon_{CU}}{2\alpha\sigma_{CB}} + \left\{\frac{P_T E_S \varepsilon_{CU}}{2\alpha\sigma_{CB}}\left(2 + \frac{P_T E_S \varepsilon_{CU}}{2\alpha\sigma_{CB}}\right)\right\}^{0.5} \quad (7)$$

$$M_{ur} = \alpha k b d_1^2 \sigma_{CB}(1-\beta k) \quad (8)$$

where ε_{CU} is the ultimate strain, and d_1 the effective height.

For double reinforcement, the following three equations are used considering stress σ_{SC} acting on the reinforcement under compression.

Case I'; $\sigma_{ST} = \sigma_{SY}$, $\sigma_{SC} < \sigma_{SY}$

$$k = \frac{P_T\sigma_{ST} - P_C E_S \varepsilon_{CU}}{2\alpha\sigma_{CB}} + \left\{\left(\frac{P_T\sigma_{ST} - P_C E_S \varepsilon_{CU}}{\alpha\sigma_{CB}}\right)^2 + \frac{\mu P_C E_S \varepsilon_{CU}}{\alpha\sigma_{CB}}\right\}^{0.5} \quad (9)$$

$$M_{ur} = bd_1^2\left\{\alpha k \sigma_{CB}(1-\beta k) + P_C E_S \varepsilon_{CU}\frac{(k-\mu)(1-\mu)}{k}\right\} \quad (10)$$

Case II': $\sigma_{ST} = \sigma_{SY}$, $\sigma_{SC} = \sigma_{SY}$

$$k = \frac{P_T\sigma_{ST} - P_C\sigma_{SC}}{\alpha\sigma_{CBZ}} \quad (11)$$

$$M_{ur} = bd_1^2\left\{\alpha k\sigma_{CB}(1-\beta k) + P_C\sigma_{SC}(1-\mu)\right\} \quad (12)$$

Case III': $\sigma_{ST} < \sigma_{SY}$, $\sigma_{SC} = \sigma_{SY}$

$$k = -\frac{P_T E_S \varepsilon_{CU} + P_C\sigma_{SC}}{2\alpha\sigma_{CB}} + \left\{\left(\frac{P_T E_S \varepsilon_{CU} + P_C\sigma_{SC}}{2\alpha\sigma_{CB}}\right)^2 + \frac{P_T E_S \varepsilon_{CU}}{\alpha\sigma_{CB}}\right\}^{0.5} \quad (13)$$

$$M_{ur} = bd_1^2\left\{\alpha k\sigma_{CB}(1-\beta k) + P_C\sigma_{SC}(1-\mu)\right\} \quad (14)$$

where $\alpha = 0.8095$, $\beta = 0.4160$, $\mu = e/d_1$

Methods of calculating the deflection of a reinforced concrete beam under short-term loading largely consist of methods which consider the elastic load jut before cracking and those which cover the period from the start of cracking up until the level where the reinforcement yield stress is reached. For reference, the following three major calculations are typically adopted for the calculation of delfection after cracking[4),5),6),7)]:

$$ACI: W = \frac{\beta M_a L^2}{EI_e} \tag{15}$$

$$I_e = \left(\frac{M_C}{M_a}\right)^3 I_G + \left\{1 - \left(\frac{M_C}{M_a}\right)^3\right\} I_{Cr} \tag{16}$$

$$Branson: W = \frac{\beta M_a L^2}{EI_e}$$

$$I_e = \left(\frac{M_C}{M_a}\right)^4 I_G + \left\{1 - \left(\frac{M_C}{M_a}\right)^4\right\} I_{Cr} \tag{17}$$

$$CEB - FIP: W = \beta L^2 \left(\frac{M_C}{E_C I_G} + \frac{4M_{C1}}{3E_C I_{Cr}}\right)$$

$$\leqslant \beta L^2 \frac{M_a}{E_C I_{Cr}} \tag{18}$$

where, bending moment just before cracking:

$[M_c = 0.9\,(\sigma_{CB})^{2/3} I_G/(h - y_2)]$

M_a: maximum bending moment after cracking

L: effective span

β: deflection coefficient (5/48 in the case of an evenly distributed load and 1/12 where the load is concentrated) $M_{c1} = M_a - M_c$.

The sectional secondary moment before and after cracking can also be given by:

Before cracking (elastic load method)

$$y_1 = \frac{(bh^2/2) + n(A_{ST} \times d_1 + A_{SC} \times e')}{bh + n(A_{ST} + A_{SC})} \tag{19}$$

$$I_G = \left(\frac{b}{3}\right)(y_1^3 + y_2^3) + n\left\{A_{ST}(d_1 - y_1)^2 + A_{SC}(y_1 - e')^2\right\} \tag{20}$$

After cracking (from cracking to the reinforcement yield point)

$$y_1 = -\left\{\frac{n(A_{ST} + A_{SC})}{b}\right\} + \left[\left\{\frac{n(A_{ST} + A_{SC})}{b}\right\}^2 + \left(\frac{2n}{b}\right)(A_{ST}d_1 + A_{SC}e')\right]^{0.5} \tag{21}$$

$$I_{Cr} = \frac{by_1^3}{3} + nA_{SC}(y_1 - e')^2 + nA_{ST}(d_1 - y_1)^2 \tag{22}$$

3. DUFFING STRESS-STRAIN RELATIONSHIP

In cases where the stress-strain relationship for structural materials is non-linear, many mathematical models have been used to express the non-linearity of the curves. In modeling such stress-strain curves, the choice of approximation is very important and depends on the purpose of the structural analysis and the required accuracy. In a general discussion of structural analysis, expressional models as close as possible to the actual stress-strain relationship may sometimes be required. In either case, straight-forward and accurate matehematical models are likely to be better accepted.

Let us consider the relationship between stress and strain as a curve. That is, the relationship between strain ε and stress σ is $\sigma = f(\varepsilon)$. Here, ε is very small, and thus σ can be expressed as a Maclaurin expansion of σ.

$$\sigma = f(0)\varepsilon + f'(0)\varepsilon + \frac{\varepsilon^2}{2!}f''(0) + \frac{\varepsilon^3}{3!}f'''(0) + \ldots \tag{23}$$

When no external force is acting, the stress is zero. Thus, $f(0) = 0$. This function $f(\varepsilon)$ is assumed to have reversibility. Then, assuming that the rupture strain is sufficiently small and that terms up to ε^3 are considered, equation (23) can be expressed as a Duffing spring.

$$\sigma = f'(0)\varepsilon + \frac{\varepsilon^3}{3!}f'''(0) = A\varepsilon - B\varepsilon^3 \tag{24}$$

Constants A and B in equation (24) are determined as follows.

First, constant A is assumed to be the elastic coefficient E. Then, the range of strain within which a recovery reaction in the component parts of a member exists is considered to be ε_B. Eventually, the relation constant $B = E/3\varepsilon_B$ is obtained from $\partial\sigma/\partial\varepsilon = 0$. Therefore, the relationship between stress σ and strain ε can be given as follows:

$$\sigma = E(\varepsilon - \frac{\varepsilon^3}{3\varepsilon_B^2}) \quad (25)$$

Supposing the restoring force f is expressed by the 1st and the 3rd polynominals of the displacement x;

$$f = \alpha x + \beta x^3 \quad (25.a)$$

where, α,β are the spring constants. It leads to the forced non-linear vibration as follows:

$$m\frac{d^2x}{dt^2} + \alpha x + \beta x^3 = F_0 \cos(\omega t) \quad (25.b)$$

where, F_0: amplitude of the input force

ω: angular velocity of the input force

m: mass of the vibrator

In equation (25), the spring force is expressed by the 1st and 3rd polynomials of displacement. In this study the stress is related to the 1st and 3rd polynomials of the strain, hence it is name as Duffing type.

When a member has a sectional profile (b X h), it is considered to have a surface bearing. Thus, moment M is obtained as follows from the relationship expressed in equation (25).

$$M = \int_{-h/2}^{h/2} \sigma y dy = \frac{Ebh^2}{6}(\varepsilon_U - \frac{\varepsilon_U^3}{5\varepsilon_B^2}) \quad (26)$$

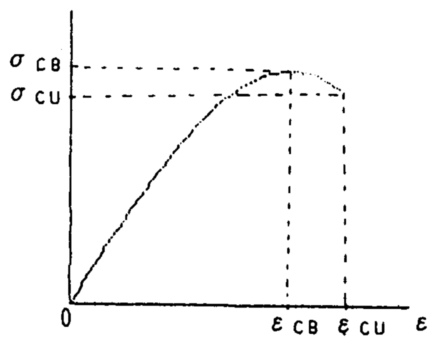


Fig. 1. Stress-strain relationship of concrete.

where, y: distance from neutral axis ($\varepsilon h/\varepsilon_U$)

ε_U: edge strain of the side under com-compression

The upper edge strain at the ultimate value of M is $dM/d\varepsilon_U = 0$, which can be derived from equation (26), while ε_U can be expressed as follows:

$$\varepsilon_U = (\frac{5}{3})^{1/2} \varepsilon_B \quad (27)$$

This value of ε_U is a form of critical compressive strain.

Then the relationships expressed by equations (25) and (27) give the stress-strain curve for the the concrete, as shown in Figure 1. The stress-strain relationship of Figure 2 is used in the JCI Concrete Specifications (3.3.3), which are used as the basis for structural analysis.

4. COMPARISON BETWEEN STRESS-STRAIN RELATIONSHIP OF CONCRETE AND THE DUFFING MODEL

Strictly speaking, the stress-strain relationship for a concrete structure is non-linear from the very beginning of loading. However, a structural member is defined as a linear elastic body during the initial stages of loading, and its elastic modulus varies with compressive strength σ_{CB}. Researchers from Japan and abroad have proposed several equations to express the relationship, in concrete, between elastic modulus and

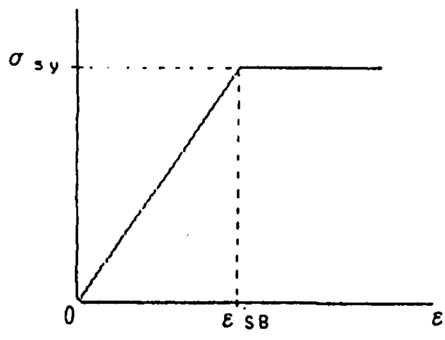


Fig. 2. Stress-strain relationship of reinforcement.

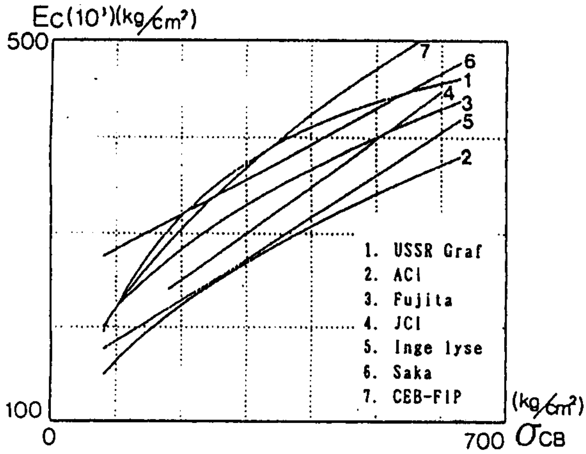


Fig. 3. Comparison of equations for E_C.

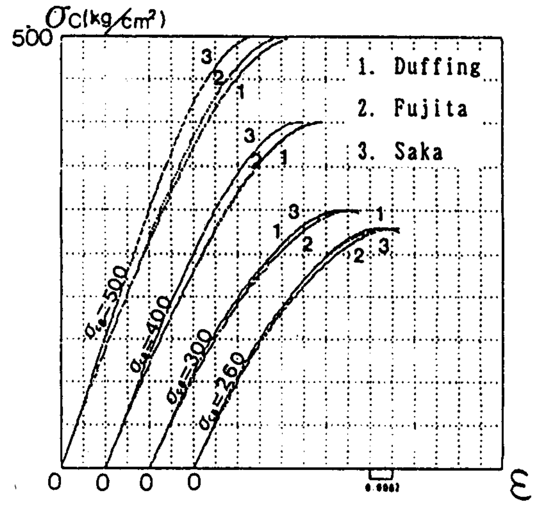


Fig. 4. Stress-strain relationship of concrete.

strength. A comparison of these proposals is given in Figure 3. The JCI value can be seen to lie almost in the middle. For this reason, the value E_C proposed by JCI is taken as the elastic modulus to be used in the Duffing equation in this study. Vaious experimental equations for the concrete's stress-strain curve are also proposed. In the study, the two cubic equations proposed by Saka and Fujita are chosen for comparison. The results are illustrated in Figure 4. The difference, as seen in these figures, is very small. Thus, the Duffing equation adequately describes the concrete's stress-strain curve.

5. REINFORCED CONCRETE SUBJECT TO BENDING FORCE

Figure 5 shows the dimensions of a reinforced concrete beam and the stress distribution within it. The horizontal balance conditions over a cross section before cracking (from states I to II: Type 1), $\Sigma H = T_C + T_S - C_C - C_S = 0$, gives the balance of the beam profile as follows:

$$\int_0^{1/2} \sigma_{C2} b dy + E_S A_{ST} \psi (h - y_1 - e)$$

$$+ \int_{-y1}^{0} \sigma_{C1} b dy + E_S A_{SC} \psi (y_1 - e') = 0 \qquad (28)$$

Where, ψ. curvature, $\varepsilon = \psi y$, y: distance from neutral axis, A_{ST}: total area of reinforcement

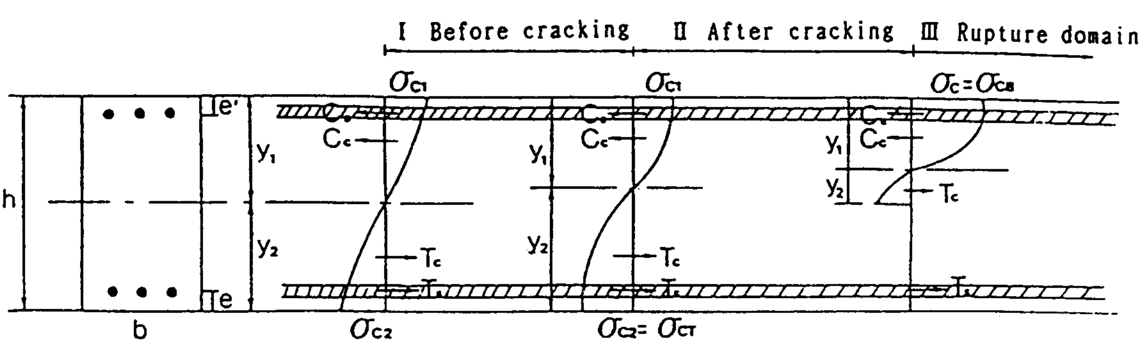


Fig. 5. Stress distribution in beam.

under tension and, A_{SC}: total area of reinforcement under compression. Given that $Z = \frac{\psi y_1}{\varepsilon_{CB}} = \frac{\varepsilon_{CC}}{\varepsilon_{CB}}$, $Z' = \frac{\psi y_2}{\varepsilon_{CB}} = \frac{\varepsilon_{CT}}{\varepsilon_{CB}}$, $Z + Z' = \frac{\psi h}{\varepsilon_{CB}}$, the following equation, expressed as a cubic equation of Z', is obtained by substituting σ_c from equation (28) into equation (25):

$$\frac{Z'^3}{12} - \frac{ZZ'^2}{12} + Z'\{\frac{Z^2}{12} - \frac{1}{2} - n(P_C\mu' + P_T - P_T\mu)\} + \frac{Z}{2} - \frac{Z^3}{12} + nZ(P_C - P_C\mu' + P_T\mu) = 0 \tag{29}$$

Where $n = E_S/E_C$, $e' = h\mu'$, $Z' \leqslant \varepsilon'_{CT}/\varepsilon_{CB}$, neutral axis $y_1 = hZ/(Z + Z')$, curvature $\psi = (Z + Z')\varepsilon_{CB}/H$, and ε'_{CT}: maximum tensile strain of the concrete.

Since the resistance offered through internal forces must equal to the bending moment M exerted by the external force, the balance of oments for the neutral axis between states I and II is expressed as:

$$M = \int_0^{1/2} E_C b(\psi y - \frac{\psi^3 y^3}{3\varepsilon_{CB}^2})y dy + E_S A_{ST}\psi(h - y_1 - e)^2 + \int_{-y_1}^{0} E_C b(\psi y - \frac{\psi^3 y^3}{3\varepsilon_{CB}^2})y dy + E_S A_{ST}\psi(y_1 - e')^2 \tag{30}$$

Also the following equation is given by manipulating the relations discussed earlier:

$$M' = \frac{M}{E_C b h^2 \varepsilon_{CB}} = \frac{Z^3 + Z'^3 - (Z^5 + Z'^5)/5}{3(Z+Z')^2} + \frac{nP_C\{Z - (Z+Z')\mu'\}^2}{Z+Z'} + \frac{nP_T(Z' - Z\mu - Z'\mu)^2}{Z+Z'} \tag{31}$$

For states II to III (cracking of Type 2 occurs), the stress in the reinforcement is $\sigma_{ST} < \sigma_{SY}$, $\sigma_{SC} < \sigma_{SY}$, $\sigma_{CT} = \sigma'_{CT}$, so the following equations can be obtained:

$$2R^2n\{-P_C\mu' - P_T(1-\mu)\} + 2RnZ(P_C + P_T) + Z^2 - Z'^2 - (Z^4 - Z'^4)/6 = 0 \tag{32}$$

$$M' = \frac{Z^3 + Z'^3 - (Z^5 + Z'^5)/5}{3R^2} + \frac{nP_C(Z - R\mu')^2}{R} + \frac{nP_T\{R(1-\mu) - Z\}^2}{R} \tag{33}$$

where $R = Z + Z' = \psi h/\varepsilon_{CB}$, $Z' = \varepsilon'_{CT}/\varepsilon_{CB}$ (constant).

Although the yield stress is reached for tensile reinforcement, compressive reinforcement as the ultimate state is approached, is in a state of $\sigma_{ST} = \sigma_{SY}$, $\sigma_{SC} < \sigma_{SY}$, $\sigma_{CT} = \sigma'_{CT}$, (Type 3), and no yielding occurs right up to the ultimate state. Also, the second term of equation (28) is $\sigma_{SY}A_{ST}$, leading to the following balance:

$$-2R^2nP_C\mu' + 2R(nP_CZ - P_T\eta) + Z^2 - Z'^2 - (Z^4 - Z'^4)/6 = 0 \tag{34}$$

$$M' = \frac{Z^3 + Z'^3 - (Z^5 + Z'^5)/5}{3R^2} + \frac{nP_C(Z - R\mu')^2}{R} + \frac{\eta P_T\{R(1-\mu) - Z\}}{R} \tag{35}$$

When the tensile reinforcement and compressive one reaches the yield point, the second and fourth terms of equation (28) in $\sigma_{ST} = \sigma_{SY}$, $\sigma_{SC} = \sigma_{SY}$, $\sigma_{CT} = \sigma'_{CT}$, (Type 4) are $\sigma_{SY}A_{ST}$, and $\sigma_{SY}A_{SC}$, respectively, resulting in the following values for R and M' from the balance equation:

$$R = \frac{E_C\varepsilon_{CB}\{Z^2 - Z'^2 - (Z^4 - Z'^4)/6\}}{2(P_T\sigma_{ST} - P_C\sigma_{SC})} \tag{36}$$

$$M' = \frac{Z^3 + Z'^3 - (Z^5 + Z'^5)/5}{3R^2} + \frac{\eta' P_C(Z - R\mu')}{R} + \frac{\eta P_T\{(R(1-\mu) - Z\}}{R} \tag{37}$$

where $\eta = \sigma_{ST}/E_C\varepsilon_{CB}$, $\eta' = \sigma_{SC}/E_C\varepsilon_{CB}$

When the compressive reinforcement yields before the tensile reinforcement, the fourth term of equation (28) in $\sigma_{ST} < \sigma_{SY}$, $\sigma_{SC} = \sigma_{SY}$, $\sigma_{CT} = \sigma'_{CT}$, (Type 5) is $\sigma_{SY}A_{SC}$, thus given the following balance equation:

$$-2R^2nP_T(1-\mu)+2R(\eta' P_C+nP_TZ)+Z^2-Z'^2$$
$$-(Z^4-Z'^4)/6=0 \qquad (38)$$

$$M'=\frac{Z^3+Z'^3-(Z^5+Z'^5)/5}{3R^2}+\frac{\eta' P_C(Z-R\mu')}{R}$$
$$+\frac{nP_T\{R(1-\mu)-Z\}^2}{R} \qquad (39)$$

In the case of single reinforcement, P_C is taken as zero according to the analysis described in Section 5.

6. NATURE ANALYSIS OF REINFORCED CONCRETE SUBJECT TO BENDING FORCE

When a reinforced concrete beam is subject to a bending force, the strain ε_{CC} on the compression side increases from zero to ε_{CU}. No cracking occurs on the tension side for strains such that $\varepsilon_{CT} < \varepsilon'_{CT}$. However, when the load rises from Type 1 to reach $\varepsilon_{CT} = \varepsilon'_{CT}$, the influence on the member changes dramatically from that point on up to the yield point. The process from cracking to yielding of the reinforcement under compression and tension will be analyzed using equations of Type 2, while the range from cracking to the ultimate state will be analyzed using equations of either type 3, 4, or 5.

In the case of single reinforcement, Type 3 is equivalent to Type 4. Calculations will be done using Type 2, 3, or 5, depending on which has a smaller R value. Also, in the case of double reinforcement, calculations will be done using either of Type 2, 3, 4, or 5, whichever has a smaller R value.

As shown in Figure 6, the values of neutral axis ratio given by our theoretical equations are almost the same as those given by the generally accepted theorem. Curvature can also be obtained from the bending strain. The results of doing this are shown in Figure 8. The relationship between reinforcement ratio and moment just before cracking is almost the same when obtained using our theoretical equation as when Fujita's equation is used. In the ultimate state, the results given by our equation differ little those obtained by the generally accepted equation when the reinforcement ratio is 0.08 or less, while as the reinforcement ratio increases, the difference widens. However, since such conditions do not exist in actual designs, our theory proves to be appropriate.

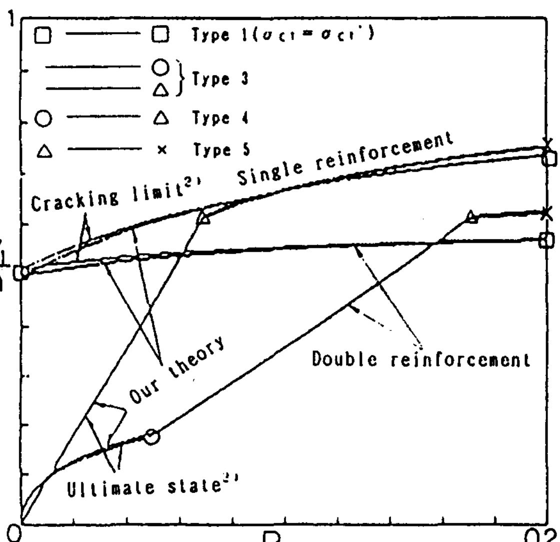


Fig. 6. Relationship between reinforcement ratio and neutral axis.

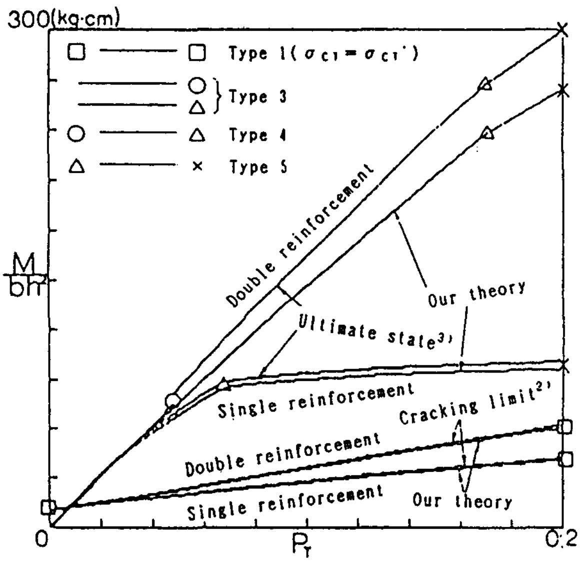


Fig. 7. Relationship between reinforcement ratio and moment.

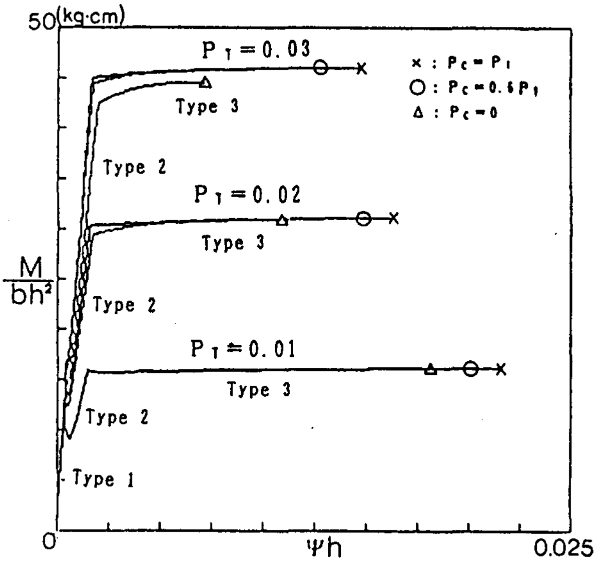


Fig. 8. Relationship between curvature and moment.

7. CALCULATION OF DEFLECTION BY FINITE INTEGRATION TRANSFORMS

The orthogonality of a since function by finite integration method is given as follows:

If γ is an variable ranging from zero to s, then any function including γ makes secnse only when γ takes the prescribed integer. The integration is shown as:

$$\int_0^s \sin\frac{i\pi r}{s}\sin\frac{m\pi r}{s}\Delta x = \xi \quad (40)$$

where, $i, m = 1, 2, 3, ..., S$. Taking $\Delta x = \xi$

$$\xi = \frac{1}{4}[\frac{\sin\pi(i-m)}{\sin\pi/s(i-m)}\{1+\cos\frac{\pi(i-m)}{s}\} - \cos\pi(i-m)+\cos\pi(i+m)] \quad (41)$$

So that

$$\xi = \begin{cases} s/2 & i=m \\ 0 & i\neq m \end{cases} \quad (42)$$

The inversion formula is established as follows;

$$\sum_{r=0}^{s-1} f(x)\sin\frac{i\pi r}{s} = \phi(i) \quad (43)$$

$$\frac{2}{s}\sum_{i=0}^{s-1}\phi(i)\sin\frac{i\pi r}{s} = f(x) \quad (44)$$

Disregarding the deflection of the beam due to shearing force, it is allowed so long as h/L is smaller than 0.1. The deflection W is related to the curvature ψ as follows:

$$\frac{d^2W}{dx^2} = -\psi \quad (45)$$

where h is the largest size of hthe beam cross section and L is the span length of the beam. Considering that W is a continuous function of x for $0 \leqslant x \leqslant l$, equation (45) can be transformed into a difference equation:

$$\frac{W_{r+1}-2W_r+W_{r-1}}{(\Delta x)^2} = -\psi_r \quad (46)$$

where $\Delta x = l/s$, s is the division number of l, W_r. deflection of r-th point, and ψ_r: curvature of r-th point. If ψ_r is given, W_r can be calculated by the above equation by means of "finite integration transforms."

The deflection, whether elastic or non-elastic, is conventionally analzed using the elastic load method. However, the distribution of curvature along the axis of a reinforced concrete beam is not simple, and thus the conventional method leads to inaccurate analysis. We now propose a method of deflection analysis based on the curvature in the casse when a simply supported reinforced concrete beam is subject to a bending force. Firstly, the curvature ratio ψ at a given point X on a reinforced concrete beam is obtained using the theoretical expressions discussed in Section 5 and 6. If the span L of a simple reinforced concrete beam is equally divided into S section and the deflection at the r th point is W_r, then equation (47) can be obtained from central difference calculus. Here, m is defined as (d^2W/dX^2).

$$\frac{dW_r}{dX} = \frac{W_{r+1}-W_r}{\Delta X}$$

$$\frac{d^2W_r}{dX^2} = \frac{W_{r+1}-2W_r+W_{r-1}}{(\Delta X)^2} = -m_r\psi_0 \quad (47)$$

where $\Delta X = L/S$, $m_r = \psi/\psi_0$, ψ_0: maximum curvature.

The theorem of Finite Integration Transform gives the conversion equations (48) and (49).

$$\sum_{r=1}^{S-1} W_r \sin\frac{i\pi r}{S} = \phi_i \tag{48}$$

$$\frac{2}{S}\sum_{i=1}^{S-1} \phi_i \sin\frac{i\pi r}{S} = W_r \tag{49}$$

where i and S are positive integers.

When function W_r in the above equations is replaced by curvature m_r,

$$\sum_{r=1}^{S-1} m_r \sin\frac{i\pi r}{S} = \phi_i \tag{50}$$

$$\frac{2}{S}\sum_{i=1}^{S-1} \phi_i \sin\frac{i\pi r}{S} = m_r \tag{51}$$

Replacement of ϕ_i by W_i' completes the conversion equations (52) and (53).

$$W_i' = \sum_{r=1}^{S-1} W_r \sin\frac{i\pi r}{S} \tag{52}$$

$$W_r = \frac{2}{S}\sum_{i=1}^{S-1} W_i' \sin\frac{i\pi r}{S} \tag{53}$$

Here, substituting equation (53) equation (47) gives equation (54), as follows:

$$\begin{aligned} &W_{r+1} - 2W_r + W_{r-1} \\ &= \frac{2}{S}\sum_{i=1}^{S-1} W_i'\{\sin\frac{i\pi(r+1)}{S} - 2\sin\frac{i\pi r}{S} \\ &\quad + \sin\frac{i\pi(r-1)}{S}\} \\ &= \frac{2}{S}\sum_{i=1}^{S-1} W_i'\{2(\cos\frac{i\pi}{S} - 1)\}\sin\frac{i\pi r}{S} \end{aligned} \tag{54}$$

By letting $D_i = 2\{1 - \cos(i\pi/S)\}$, equation (49) yields the following equation with the aid of equations (17) and (54).

$$\frac{2}{S}\sum_{i=1}^{S-1} W_i' D_i \sin\frac{i\pi r}{S} = (\Delta X)^2 m_r \tag{55}$$

Substituting equation (51) for m_r in the above equation gives equation (56).

$$\frac{2}{S}\sum_{i=1}^{S-1} W_i' D_i \sin\frac{i\pi r}{S} = (\Delta X)^2 \frac{2}{S}\sum_{i=1}^{S-1} \phi_i \sin\frac{i\pi r}{S} \tag{56}$$

$$W_i' = (\Delta X)^2 \frac{\phi_i}{D_i} \tag{57}$$

Also substituting equation (50) and (53) for ϕ_i and W_i' in equation (57) finally leads to equation (58).

$$W_r = \frac{2}{S}\sum_{i=1}^{S-1} (\Delta X)^2 (\sum_{r=1}^{S-1} m_r \sin\frac{i\pi r}{S}) \sin\frac{i\pi r}{SD_i} \tag{58}$$

which assures us to calculate the deflection of the beam by the curvature value of m_r.

8. EXAMPLE OF DUFFING ANALYSIS OF NON-LINEAR BEAM

If we take the mechanical characteristics of materials comprising a beam to be non-linear-that is, if the mechanical characteristics of component materials, such as composite materials, and cracking are not considered, an analytical model with sectional profile width b and height h is used for analysis. First, the moment M when the stress-strain relation follows equation (25) is expressed as:

$$M = \int_A \sigma y dA = E(I_2\psi - \frac{I_4\psi^3}{3\varepsilon_B^2}) \tag{59.a}$$

$$M' = \frac{M}{EI_2\psi_0} = \frac{\psi}{\psi_0} - \frac{I_4\psi^3}{3I_2\psi_0\varepsilon_B^2} \tag{59.b}$$

where $I_2 = bh^3/12, I_4 = bh^5/80$, and maximum curvature $\psi_0 = 2\varepsilon_B/h$.

Assuming that m $= \psi/\psi_0$ leads to the following equation:

$$M' = m - \frac{m^3}{5} \tag{59}$$

As indicated by this equation, if the bending moment when the non-linear curvature ratio m equals 1 is the bending moment when stress is maximum, $m' = 4/5$ and for the linear case, $M' = 1$.

Next, supposing the analytical model is such that load is equally distributed over a simple beam of span $L = 5m$, the relationship between the moment M' at a given point X and curvature can be expressed as:

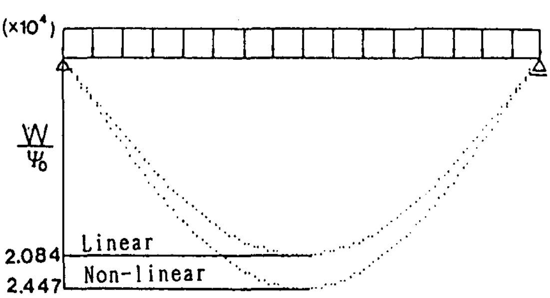
$$M' = \frac{M}{EI_2\psi_0} = \frac{4}{5} = \frac{qL^2}{8} \qquad q = \frac{32}{5L^2} \quad (60)$$

That is, the moment M' at any point X will be:

$$M' = \frac{qX(L-X)}{2EI_2\psi_0} = \frac{16X(L-X)}{5L^2EI_2\psi_0} \quad (61)$$

If equation (60) is linear for the moment M' at a given point X on the beam, the cubic term need not be considered. If it is non-linear, the cubic term must be considered when calculating the curvature ratio m. The deflection in both linear and non-linear cases can be obtained by replacing m as defined above with m_r in equation (58) for the finite intergration. The results thus obtained are shown in Figure 9. These values are deflection W divided by maximum curvature ψ_0.

Taking a case of $b = 35cm$, $h = 70cm$, $\sigma_B = 300kg/cm^2$, and $E = 3 \times 10^5 kg/cm^2$, for example, $\varepsilon_B = 0.0015$ according to the relation expressed by equation (25). That is the maximum load $q_{max} = 329.28kg/cm$, $\psi_0 = 2\varepsilon_B/h = 4.2875 \times 10^{-5}$. Table 1 gives a comparison between central deflection for each load increment up to the maximum load and the results obtained by the elastic load method.



Figur 9. Deflection.

Table 1 shows that the linear version of the finite integration transforms calculus and the deflection obtained by the elastic load method are almost equal. That is, the results obtained by the inversion formula (58) is sufficiently accurate. As the load increases towards the ultimate value, the comparision of the linear and non-linear results indicates that a difference in curvature and deflection arises more between them.

For a reinforced concrete beam, the curvature along the axis is complicated by the interaction between reinforcement and concrete. It cannot be determined just by considering the elastic load or using equation (60), a non-linear curvature ratio equation. Therefore, the bending moment and curvature of a reinforced concrete beam as dicussed in Section 5 is analyzed to obtain the relationship between any given bending moment and the resulting curvature.

Let us use as an analytical model a simple beam with a span $L = 5m$, $\sigma_{CB} = 300kg/cm^2$, $E_C = 3.0 \times 10^5 kg/cm^2$, $\sigma_{SY} = 2400kg/cm^2$, and analyze it for two cases: single reinforcement ($P_T = 0.01$) and double reinforcement ($P_C = 0.6P_T$). The curvature ψh can be obtained using the method discussed in Section 5. Then from Figure 8, representing the relationship between moment and curvature, the curvature at any given point on the beam can be found as illustrated in Figure 10. Next, substituting curvature

Table 1. Comparison of Deflection by Two Analysis Methods (in cm).

Load Analysis method	60%	70%	80%	90% kg/cm	329.28 kg/cm
Elastic load method	0.5357	0.6250	0.7143	0.8036	0.8928
Finite Integration Transforms linear equation	0.5359	0.6252	0.7145	0.8038	0.8931
Finite Integration Transforms non-linear equation	0.5582	0.6629	0.7753	0.9001	1.0486

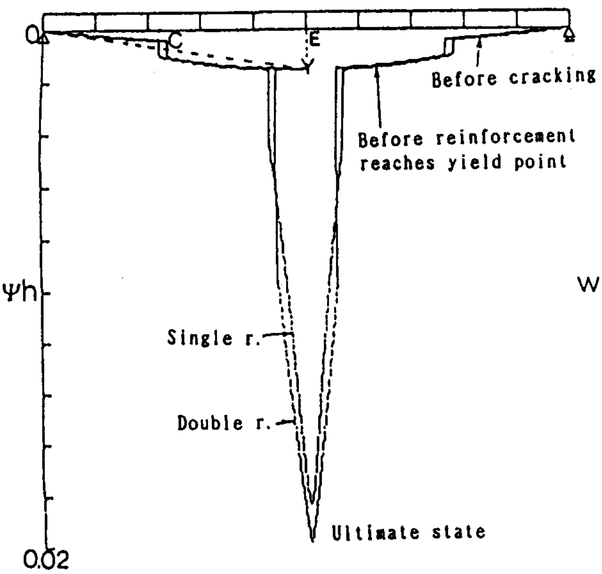


Figure 10. Reinforced concrete curvature.

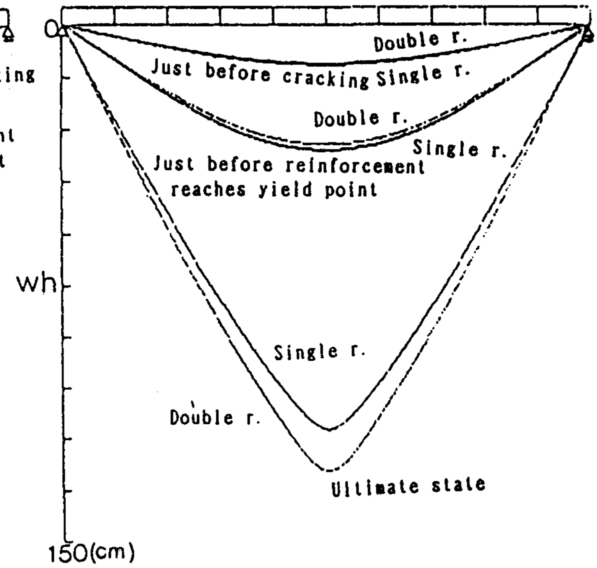


Figure 11. Reinforced concrete deflection.

ψh for m_r in equation (58) gives the deflection at any given point, as shown in Figure 1. M_{max}/bh^2 due to the inter-span central bending moment for a beam with a profile $b = 35cm$ and $h = 70cm$ just before the reinforcement yields is $15.798kg/cm^2$ for single reinforcement and $15.940kg/cm^2$ for double reinforcement, as shown in Figure 8. The maximum deflection just before the reinforcement yields when the sectional secondary moment is equal to the pre-cracking design bending moment M_C, as expressed in ACI equation (16), Branson's equation (17), and CEB-FIP equation (18), is compared with the theoretical values obtained in our study in Table 2. The ACI equation supposes that cross-sectional rigidity is constant all along the beam's length, while Branson's equation uses a value of sectional rigidity which varies according to the bending moment. The CEB-FIP equation obtains regidity by dividing the situation into pre-cracking and post-cracking states. This means that the rigidity along the dotted line OYE in Figure 10 is obtained by a conversion. In this case, for an evenly distributed load, the difference in area between the dotted line OYE and theline OCDYE is small, and thus the error in deflection from our theory becomes small. However, when a concentrated load is placed at the center of the span, the difference between the dotted line OYE and the line OCDYE widens, and eventually error in deflection value increases. When the reinforcement collapses, yielding of the reinforcement leads to a rapid increase in curvature. At this point, conventional

Table 2. Comparison of Maximum Deflection before Yield by Three Analysis Methods (in: cm)

	Single reinforcement ($P_T = 0.01$)				Double reinforcement ($P_C = 0.6P_T$)			
Analysis method	ACI	CEB-FIP	Branson	Our theory	ACI	CED-FIP	Branson	Our theory
Concentrated load	0.37427	0.38685	0.41171	0.34525	0.35216	0.36491	0.39104	0.32234
Evenly distributed	0.46784	0.48357	0.51464	0.51312	0.44020	0.45613	0.48880	0.48883

deflection equations cannot be used. However, as shown in Figure 10, our theory allows a curvature distribution for all deformation conditions of the beam, from the elastic domain to rupture, to be obtained. Thus, equation (58) can be used to obtain the deflection all the way from the elastic state to the ultimate state.

9. CONCLUSION

Appling the relationship that stress in concrete is expressed by the 1st and the 3rd polynomials of the strain, ie., duffing type stress strain relation, this study obtained a relation between curvature and the bending moment at any section of a reinforced concrete beam ranging from elastic to non-elastic states. The curvatures lead to the corresponding deflection is calculated using equation (58), which is derived by finite integration transforms shown in paragraph 7. The deflection corresponding to any stress state of the simple R.C. beam can thus by calculated. The important findings of this study are listed as follows:

1. Figure 4 shows that Duffing stress-strain relation well agrees to the results so far given by Fujita and Saka.
2. On the reinforced concrete beam under bending force, the variations of reinforcement ratio with neutral axis ratio, of reinforcement ratio with bending moment and of curvature with bending moment which are computed by he presenting theory are found to be in good agreement to those derived from the conventional analyses.
3. Table 1 confirms that the deflection equation based on the finite integration transforms is sufficiently correct. That is, the sufficient deflection can be obtained when the relation between bending moment and curvature at any section is revealed.
4. Equilibrium of force at any section of the reinforced concrete beam, taking Duffing non-linearty into account, leads to the curvature of the section. Thus obtaind curvatures yield the deflections by means of the finite integration transforms as shown in Figure 10, 11, and Table 2. It can give the deflection in the ultimate state. Incidentially, the division number S in equation (58) becomes larger, so does the accruacy higher.

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