# 專 論

# Numerical Analysis of Water Infiltration*

# 滲透之數式解法

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## ABSTRACT

In attempting to numerically solve the nonlinear moisture flow equation, the Galerkin process, which bears a great similarity to direct methods of the calculus of variations, and the CSMP (Continuous System Modeling Program) approach have been employed. In the finite element formulation of the governing equation, systems of nonlinear algebraic equations were developed on the basis of linear two-dimensional triangular elements. These nonlinear algebraic equations were solved simultaneously, at each time step, by a programmed logic of iterations. In the CSMP approach, Boltzmann's function application and layered soils formulation and layered soils formulation wer edemonstr ated in obtaining vertical moisture profiles.

## 摘 要

土壤水份運動並非綫性公式，在此將以兩種方法解之。其一爲應用 Galerkin 氏法，其二爲利用 IBM CSMP 的已有程式。前者終極爲解決非綫性聯立方程式，後者則無此問題，且可應用於有層次之土壤剖面。

## INTRODUCTION

The recent increased demand in urban development, hydrological simulation, irrigation systems design, and water resources management, etc., have reemphasized the need for better analysis and predictive methods in the rainfall-runoff dynamic process, which is the time variate of the process being considered. The due consideration of the time-dependent infiltration into soil is one of the important components in a sensitive and effective prediction of runoff from precipitation.

The nonlinear partial differential equations delineating soil-moisturerelations are obtained by using dynamic equations, which interrelated the energy dissipated in the system with the velocity, i.e., Darcy's law, induced ina conservation of mass statement yielding the desired basic flow equation for homogeneous isotropic media.

* 本文爲 1975 年 11 月 16 日下午在本會 64 年度年會之學術講演。

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In the early stages, the Boltzmann transformation was applied by Philip (1957) and Gardner and Mayhugh (1958) to reach a solution of the nonlinear partial differential equation by numerical integration. Later, the finite difference method was used by Ashcroft et al. (1962) in an attempt to solve the flow equation. Under the same concept of numerical solution, i.e., finite difference method, Freeze (1969) developed a method to account for many of the processes occurring under field conditions. but no comparison was made with experimental data. In late 1969, Hanks,Klute, and Bresler established a generalized numerical method (finite difference approach) of infiltration redistribution, drainage, and evaporation.

In the past several years, an alternative approach, the finite element method has been introduced to the domain of fluid flow theory. The finite element method has been used successfully in the field of structural and continuum mechanics (Zienkiewicz and Cheung 1967).

Applications of the finite element method to steady porous media flow problems have been studied by Zienkiewicz, Mayer, and Cheung (1966), Taylor and Brown (1967), Neuman and Witherspoon (1970), and many others. For transient groundwater flow problems, studies have been made by Witherspoon, Javandel and Neuman (1968), Neuman and Witherspoon (1971), Neuman (1972), Bruch (1973), and many others.

In 1967, a powerful, dynamic simulation language called the Continuous System Modeling Program (CSMP) was developed by IBM which provides another approximate solution to the transient moisture flow problems. The CSMP language (IBM 1972) was presented in such a way that the programs can be understood without any prior knowledge of programming techniques. The popularity of this language has been growing rapidly in the past few years. Applications of CSMP to subsurface flow problems have beendemonstrated by Bhuiyan et al. (1971), van Keulen and van Beek (1971), de Wit and van Keulen (1972), van Keulen (1975), and many others.

## MATHEMATICAL PRELIMINARY

Darcy's equation for one-dimensional unsaturated porous media flow can be expres-sedas

$$V = -K(\theta)\frac{d\phi}{dz^1} \tag{1}$$

where V=Darcy's discharge velocity;

K=hydraulic conductivity of the medium;

$\theta$ =volumetric moisture content;

$\phi$ =total potential energy$=\frac{P}{\gamma}+z'$

(where P=hydrostatic pressure,

$\gamma$=specific weight of the fluid, and

$z'$=the vertical ordinate with positive sign upward).

The law of conservation of mass states that

$$\frac{\partial\theta}{\partial t} = \frac{\partial V}{\partial z'} \tag{2}$$

where t=time.

Substituting equation (1) into equation (2) yields

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial z'}[K(\theta)\frac{\partial\phi}{\partial z}] \tag{3}$$

Having the moisture potential $\psi$ equals the pressure head $\frac{P}{\gamma}$, let $z'$ equal $-z$, then $\phi$ becomes

$$\phi=\psi-z \tag{4}$$

where z is positive downward.

Substituting equation (4) into equation (3) yields

$$\frac{\partial\theta}{\partial t}=\frac{\partial}{\partial z}[K(\theta)\frac{\partial\psi}{\partial z}]-\frac{\partial K(\theta)}{\partial z} \tag{5}$$

By means of chain rule,

$$\frac{\partial\psi}{\partial z}=\frac{\partial\psi}{\partial\theta}\ \frac{\partial\theta}{\partial z} \tag{6}$$

And if K and $\psi$ are single-valued functions of $\theta$, then

$$D(\theta)=K(\theta)\frac{\partial\psi}{\partial\theta} \tag{7}$$

where $D(\theta)$ = moisture diffusivity of the medium.

Snbstituting equations (6) and (7) into equation (5) yields

$$\frac{\partial\theta}{\partial t}=\frac{\partial}{\partial z}[D(\theta)\frac{\partial\theta}{\partial z}]-\frac{\partial K(\theta)}{\partial z} \tag{8}$$

Equation (8) is the popular vertical infiltration formula (Philip 1957). When only the horizontal movement is involved, soil water conductivity isnot considered, since the horizontal moisture movement equation has exactly the same form as equation (8) excluding the last term on the right hand side, i.e.,

$$\frac{\partial\theta}{\partial t}=\frac{\partial}{\partial z}[D(\theta)\frac{\partial\theta}{\partial z}] \tag{8.1}$$

In an attempt to solve the nonlinear vertical infiltration equation, the variational method of the finite element technique was employed. The procedures were (1) to find the functional of the governing differential equation for a givenproblem which Euler's condition satisfied; and (2) to minimize the functional to generate a solution. In other words, solution of the governing differential equation is obtained indirectly by minimizing its functional. Unfortunately, the approach of variational formulation for nonlinear problems has no generalized means of finding its functional, for problems of this type do not always lead naturally to variational formulations (Mikhlin 1971), and little success has been experienced in that attempt (Remson, Hornberger, and Molz 1971). Consequently, research efforts were concentrated on the weighted residual method of approximation and some research findings are reported herein.

## WEIGHTED RESIDUAL METHOD OF APPROXIMATION

### Weighted Residual Method

The weighted residual method seeks an approximate solution that is close to the exact solution in the sense that the difference between residuals is minimized for most nonlinear problems (Ames 1965). Although this method does not stem from a variational principle, it bears a great similarity to direct methods of the calculus of variations, such as the Ritz method of solving boundary value problems based on reformulating the given problem as a minimization problem. Therefore, it can be used for solving nonlinear problems even when a classical variational principle does not exist (Remson, Hornberger, and Molz 1971).

In equation (8), let $g(z,t)=\frac{\partial\theta(z,t)}{\partial t}$

and $$A[\theta]=\frac{\partial}{\partial z}[D(\theta)\frac{\partial\theta}{\partial z}]-\frac{\partial K(\theta)}{\partial z}$$

where A is a differential operator.

Thus, equation (8) can be expressed as

$$g(z,t)=A[\theta] \text{ or simply } A[\theta]-g=O \quad (9)$$

Equation (9) is valid in a solution region E.

Suppose an approximate solution to equation (9) can be expressed in a linear trial-function, such as

$$\theta(z,t)=\sum_{i=1}^{n} C_i(z,t)\theta_i(z,t) \quad (10)$$

and after substituting equation (10) into equation (9), the residual R(z,t) can thus be defined as

$$R(z,t)=A(\sum_{i=1}^{n}C_i\theta_i)-g\neq O \quad (11)$$

The n parameters $C_i$ in equation (11) are determined by the n integral equations

$$\int_E W_r R(z,t)\, dE=O \quad (12)$$

For r=1,2,...,n. Where $W_r$=weighing functions, and dE=asubregion of E. If E is divided into finite number of elements, then dE will be one of the elements.

Again substituting equation (11) into equation (12) yields

$$\int_E W_r[A(\sum_{i=1}^{n}C_i\theta_i)-g]\, dE=O \quad (13)$$

for r=1,...,n.

In equation (13), $\theta_i$ can be solved by various trial-function techniques and the use of weighing functions known as weighted residual methods.

## The Galerkin Process

Consider that solution region E is divided into a finite number of triangular elements with shape functions, such as, $N_i$, $N_j$, and $N_k$ (Fig. 1), then equation (13) will be valid from element to element within region E. The Galerkin process is, simply, let $W_r=N_i$, which leads in general to the best approximation (Zienkiewicz 1971). Consequently, equation (13) becomes

$$\int_E N_i[A(\sum_{i=1}^{n}C_i\theta_i)-g]\, dE=O \quad (14)$$

Equation (14) will lead to a set of n simultaneous nonlinear algebraic equations. It is evident that the differential operator A in equation (14) will result in a higher order of differential terms and can be expressed as

$$\int_E N_i[A(\theta)-g]\, dE=O$$

i.e., $$\int_E N_i \{\frac{\partial}{\partial z}[D(\theta)\frac{\partial\theta}{\partial z}]-\frac{\partial K(\theta)}{\partial z}-\frac{\partial\theta}{\partial t}\}\, dzdt=O$$

where $dE=dzdt$,

or, $$\int_E [N_i \frac{\partial D(\theta)}{\partial z} \frac{\partial \theta}{\partial z} + N_i D(\theta) \frac{\partial^2 \theta}{\partial z^2} - N_i \frac{\partial K(\theta)}{\partial z} - N_i \frac{\partial \theta}{\partial t}] dzdt = O \qquad (15)$$

The appearance of the second term in equation (15) will severely limit the choice of shape functions. To overcome this handicap, the higher order of differential terms may be transformed by means of integration by parts. As stated by Zienkiewicz (1971):

> If this transformation is accomplished in the general form, then no restrictions are implied and if a lower order of integrals results continuity requirements of these only need to be satisfied.

## FINITE ELEMENT FORMULATION

### The Higher Order of the Differential Term

The second term of equation (15) may be transformed by using integration by parts:

$$\int udv = uv - \int vdu.$$

Consider $\int_E N_i D(\theta) \frac{\partial^2 \theta}{\partial z^2} dzdt = \int_S \{\int udv\} dt$ (16)

Let $v = \frac{\partial \theta}{\partial z}$, the $dv = \frac{\partial^2 \theta}{\partial z^2} dz$.

Let $u = N_i D(\theta)$, then $du = d[N_i D(\theta)]$

Therefore, equation (16) becomes

$$\int_E N_i D(\theta) \frac{\partial^2 \theta}{\partial z^2} dzdt = \int_S \{\int N_i D(\theta) \frac{\partial^2 \theta}{\partial z^2} dz\} dt = \int_S \{N_i D(\theta) \frac{\partial \theta}{\partial z} - \int \frac{\partial \theta}{\partial z} d[N_i D(\theta)]\} dt$$

$$= \int_S N_i D(\theta) \frac{\partial \theta}{\partial z} dt - \int_S \{\int \frac{\partial \theta}{\partial z} [N_i dD(\theta) + D(\theta) dN_i]\} dt$$

$$= \int_S N_i D(\theta) \frac{\partial \theta}{\partial z} dt - \int_S \{\int \frac{\partial \theta}{\partial z} [N_i \frac{\partial D(\theta)}{\partial z} + D(\theta) \frac{\partial N_i}{\partial z}] dz\} dt$$

$$= \int_S N_i D(\theta) \frac{\partial \theta}{\partial z} dt - \int_E [N_i \frac{\partial \theta}{\partial z} \frac{\partial D(\theta)}{\partial z} + D(\theta) \frac{\partial \theta}{\partial z} \frac{\partial N_i}{\partial z}] dzdt \qquad (17)$$

Substituting equation (17) into equation (15) yields

$$\int_E [N_i \frac{\partial \theta}{\partial z} \frac{\partial D(\theta)}{\partial z} - N_i \frac{\partial \theta}{\partial z} \frac{\partial D(\theta)}{\partial z} - D(\theta) \frac{\partial \theta}{\partial z} \frac{\partial N_i}{\partial z} - N_i \frac{\partial K(\theta)}{\partial z} - N_i \frac{\partial \theta}{\partial t}] dzdt$$

$$+ \int_S N_i D(\theta) \frac{\partial \theta}{\partial z} dt = O$$

or $$\int_E [D(\theta) \frac{\partial \theta}{\partial z} \frac{\partial N_i}{\partial z} + N_i \frac{\partial K(\theta)}{\partial z} + N_i \frac{\partial \theta}{\partial t}] dzdt - \int_S N_i D(\theta) \frac{\partial \theta}{\partial z} dt = O \qquad (18)$$

where $S$ = external surface area of solution region $E$.

### Trial-Function

An approximate solution toequation (18) was chosen in a trial-function form so that the solution would be valid from neighboring nodal points of adjacent elements, and extendsto the entire solution region. Consider the following linear trial-function:

$$\theta(z,t) = \sum_{i=1}^{n} N_i(z,t) \theta_i \qquad (19)$$

Comparing equation (19) to equation (10), then $C_i = N_i$

In general form, equation (19) will be

$$\theta(z,t)=\sum_{i=1}^{n}\sum_{j=1}^{m}N_{ij}\theta_{ij} \tag{20}$$

For triangular elements, m=3, then equation (20) appears to be

$$\theta(z,t)=\sum_{i=1}^{n}\sum_{j=1}^{3}N_{ij}\theta_{ij} \tag{21}$$

Substituting equation (21) into (18) yields

$$F(I)=\int_E [D(\theta)\frac{\partial}{\partial z}(\sum_{i=1}^{n}\sum_{j=1}^{3}N_{ij}\theta_{ij}\frac{\partial N_i}{\partial z}+N_i\frac{\partial K(\theta)}{\partial z}+N_i\frac{\partial}{\partial t}(\sum_{i=1}^{n}\sum_{j=1}^{3}N_{ij}\theta_{ij})\ dzdt$$

$$-\int_S N_i D(\theta)\frac{\partial}{\partial z}(\sum_{i=1}^{n}\sum_{j=1}^{3}N_{ij}\theta_{ij})dt=0 \tag{22}$$

Eq (22) is the Galerkin's representation of the vertical flow equation by using triangular elements.

## NaturalCoordinates and Shape Functions

A natural coordinate system is a local system (defining a particular element in the solution region) which permits the specification of a point within the element by a set of dimensionless numbers whose magnitudes never exceed unity (Desai and Abel 1972). It is precisely because of this natural coordinate system that the shape of an element is defined. For instance, the shape functions, $N_i$, $N_j$ and $N_k$, of the triangularelement, $\Delta_{ijk}$, in Figure 1 may be defined as

$$N_i=\frac{a_i+b_iz+c_it}{2\Delta},\ N_j=\frac{a_j+b_jz+c_jt}{2\Delta},\ N_k=\frac{a_k+b_kz+c_kt}{2\Delta}$$

where $\Delta$=area of $\Delta_{ijk}$

$a_i=z_jt_k-z_kt_j$ $\quad a_j=z_kt_i-z_it_k$ $\quad a_k=z_it_j-z_jt_i$

$b_i=t_j-t_k$ $\quad b_j=t_k-t_j$ $\quad b_k=t_i-t_j$

$c_i=z_k-z_j$ $\quad c_j=z_i-z_k$ $\quad c_k=z_j-z_i$

The cartesian coordinate system, also known as a rectangular cartesian coordinate system, is a system in n dimensions where n is any integer made by using n number axes intersecting each other at right angles at an origin, enabling any point within that rectangular space to be identified by the distances from the n lines. Relationship between the natural coordinate system is illustrated in (Fig. 1):

$$z=N_iz_i+N_jz_j+N_kz_k \tag{23}$$

$$t=N_it_i+N_jt_j+N_kt_k \tag{24}$$

$$N_i+N_j+N_k=1 \tag{25}$$

Consequently, for each set of $N_i$, $N_j$, and $N_k$, their corresponding set of cartesian coordinatesis unique. It is owing to equation (25) that at node i, $N_i=1$ and $N_j=N_k=0$, and on side jk, $N_i=0$, etc. A linear relation een between the natural and cartesian systems implies that contours of $N_i$, $N_j$, and $N_k$ are equally placed straight lines parallel to sides jk, ki, and ij, respectively. It is again precisely because of this relationship that in the due course of deriving systems of nonlinear algebraic equations,the last term in equation (22), i.e.,

$$\int_S N_i D(\theta) \frac{\partial}{\partial z} \left( \sum_{i=1}^{n} \sum_{j=1}^{3} N_{ij} \theta_{ij} \right) dt$$

will vanish (because $N_i = 0$ in S [Fig. 2a]).

## Simultaneous Nonlinear Algebraic Equations

The basic computer programming strategy in solving equation (22) was to consider a group of three adjacent elements at a time (Fig. 2b). As indicated in Figure 2b, moisture contents along the z and t coordinates were established, as initial and boundary conditions, respectively. A nonlinear algebraic equation containing the five nodal points (with $\theta_{11}$, $t_1$ and $\theta_{33}$, $t_1$ as unknowns) of the three considered elements can be obtained. Similarly, a set of nonlinear algebraic cquations pertaining to the first time step can thus be established by considering the second, third,..., groups of triangular elements. This procedure of setup systems of simultaneous nonlinear algebraic equations has been demonstrated by Zyvoloski and Bruch (1973). Solutions for the unknowns at each time step are provided by a programmed logic of iterations (Powell 1970).

It is worthy of note that during the process of deriving systems of nonlinear algebraic equations from equation (22), Felippa (1966) simplified the seemingly cumbrous task of integrating the polynomial terms for the triangular element formulation in equation (22).

$$\int_\Delta N_1^p N_2^q N_3^r = \frac{p!q!r!}{(p+q+r+2)!} 2\Delta$$

(where $N_1$, $N_2$, and $N_3$ are shape functions for the triangular element $\Delta_{123}$; and $\Delta$=the area of $\Delta_{123}$).

In this report, second and third degrees of polynomial interpretations of the moisture diffusivity, D, and hydraulic conductivity, K, of the media were conducted. Their respective systems of nonlinear algebraic equations were presented as follows:

For $\quad D(\theta) = D_0 + D_1\theta + D_2\theta^2, \; K(\theta) = K_0 + K_1\theta + K_2\theta^2$

Then $\quad F(I) = \sum_{i=1}^{3} (T1_i \cdot T2_i + T3_i + T4_i + T5_i + T6_i) = 0 \qquad (26)$

Where

$$T1_i = \frac{1}{4\Delta_i} [b_2^{i1}(I)\theta_{i1}(I) b_{i1}(I) b_{i2}(I) \theta_{i2}(I) + b_{i1}(I) b_{i3}(I) \theta_{i3}(I)]$$

$$T2_i = [D_0 + \frac{1}{3} D_1 \sum_{i=1}^{3} \theta_{ij}(I)]$$

$$T3_i = \frac{1}{6} D_1 [\sum_{j=1}^{3} \theta_i^{2j}(I) + \theta_{i1}(I)\theta_{i2}(I) + \theta_{i1}(I)\theta_{i3}(I) + \theta_{i2}(I)\theta_{i3}(I)]$$

$$T4_i = \frac{1}{6} K_1 [\sum_{j=1}^{3} b_{ij}(I)\theta_{ij}(I)]$$

$$T5_i = \frac{1}{6} K_2 \{b_{i1}(I)\theta_i^{21}(I) + \tfrac{1}{2} b_{i2}(I)\theta_i^{22}(I) + \tfrac{1}{2} b_{i3}(I)\theta_i^{23}(I) + [\tfrac{1}{2} b_{i1}(I) + b_{i2}(I)]\theta_{i1}(I)\theta_{i2}(I) + [\tfrac{1}{2} b_{i1}(I) + b_{i3}(I)]\theta_{i1}(I)\theta_{i3}(I) + \tfrac{1}{2} [b_{i2}(I) + b_{i3}(I)]\theta_{i2}(I)\theta_{i3}(I)]\}$$

$$T6_i = \frac{1}{6} \sum_{j=1}^{3} c_{ij}(I)\theta_{ij}(I)$$

For $\quad D(\theta) = D_0 + D_1\theta + D_2\theta^2 + D_3\theta_3, \; K(\theta) = K_0 + K_1\theta + K_2\theta^2 + K_3\theta^3$

Then $$FF(I)=\sum_{j=1}^{n}(T1_i \cdot T2_i+T3_i+T4_i+T5_i+T6_i+T7_i+T8_i)=0 \qquad (27)$$

Where $$T7_i=\frac{1}{10}D_3\{\sum_{j=1}^{3}\theta_{ij}^{3}(I)+\theta_{11}(I)[\theta_{12}^{2}(I)+\theta_{13}^{2}(I)]+\theta_{12}(I)[\theta_{11}^{2}(I)+\theta_{13}^{2}(I)]$$

$$+\theta_{13}(I)\ [\theta_{11}^{2}(I)\ +[\theta_{12}^{2}(I)+\theta_{11}(I)\theta_{12}(I)]\}$$

$$T8_i=\frac{1}{20}K_3\{2b_{i1}(I)\ \theta_{i1}^{3}(I)+\sum_{j=1}^{3}b_{ij}(I)\theta_{ij}^{3}(I)+\theta_{11}(I)\theta_{12}^{2}(I)\ [2b_{i2}(I)$$

$$+b_{11}(I)]+\theta_{11}(I)\theta_{13}^{2}(I)\ [2b_{13}(I)+b_{11}(I)]+\theta_{11}^{2}(I)\theta_{12}(I)[2b_{11}(I)$$

$$+3b_{2}^{i}(I)]+\theta_{11}^{2}(I)\theta_{13}(I)[2b_{11}(I)+3b_{13}(I)]$$

$$+[b_{i2}(I)+b_{13}(I)]\ [\theta_{2}(I)\theta_{13}^{2}(I)\ +\theta_{12}^{2}(I)\theta_{13}(I)]$$

$$+\theta_{11}(I)\theta_{12}(I)\theta_{13}(I)[b_{11}(I)+2b_{12}(I)+2b_{13}(I)]\}$$

In equations (26) and (27), the argument I=1,2,... was exhausted at the last group of elements in each time step.

## THE CSMP SOLUTION

The S/360 Continuous System Modeling Program (CSMP) which is the brainchild of IBM researchers, provides a broad spectrum of applications in simulating dynamic systems. In using the CSMP, a simulation problem is programmed for solution by preparing structure, data, and control types of statements. The input language enables a user to prepare statements describing a physical system, starting from a differential equation representation of that system. A simulation problem is then solved through commanding statements in applying the necesisary bult-in functions (total of 34 functional blocks).

The S/360 CSMP was designed for the general purpose of simulating time dependent continuous systems, and its high degree of flexibility offers almost a free hand in programming logic. A simulation problem can be programmed in almost completely different logic, depending upon approaches adopted in solving that problem. For the problem onhand, two versions of programming logic are illustrated.

The first approach is applying Boltzmann's transformation in solving the horizontal infiltration problem. Recall that equation (8.1), the horizontal moisture movement equation, can be expressed as

$$\frac{\partial\theta}{\partial t}=\frac{\partial}{\partial z}[D(\theta)\frac{\partial\theta}{\partial z}] \qquad (8.1)$$

where t and z are independent variables. For programming purposes, t and z will be substituted by Boltzmann's transformation function, as

$$\lambda(\theta)=zt^{-1/2} \qquad (28)$$

where $\lambda$=Boltzmann's variable, and $z$=distance.
Taking the partial derivative of $\lambda$ withrespect to t yields

$$\partial t=-\frac{2}{z}t^{3/2}\ \partial\lambda=-\frac{2}{z}\ \frac{t}{t^{-1/2}}\partial\lambda$$

or $$\partial t = \frac{-2t}{\lambda} \partial\lambda \qquad (29)$$

Similarly, $$\partial z = t^{1/2} \partial\lambda \qquad (30)$$

Substituting equations (29) and (30) into equation (8.1) yields

$$\frac{d\theta}{d\lambda} = -\frac{2}{\lambda} \frac{d}{d\lambda}[D(\theta)\frac{d\theta}{d\lambda}]$$

or $$\frac{d^2\theta}{d\lambda^2} = -\frac{d\theta}{d\lambda}[\frac{\lambda}{2} + \frac{dD(\theta)}{d\lambda}]\frac{1}{D(\theta)} \qquad (31)$$

In equation(31), $\frac{dD(\theta)}{d\lambda}$ can be substituted by $\frac{dD(\theta)}{d\theta}\frac{d\theta}{d\lambda}$

The final expression is

$$\frac{d^2\theta}{d\lambda^2} = -\frac{d\theta}{d\lambda}[\frac{\lambda}{2} + \frac{dD(\theta)}{d\theta}\frac{d\theta}{d\lambda}]\frac{1}{D(\theta)} \qquad (32)$$

with $\lambda$ as the only independent variable.

In CSMP solution the initial value of $d\theta/d\lambda$ in equation (32) is related to the initial value of the soil in an unknown way, as suggested by de Wit and van Keulen (1972) simulations have to be carried out for a range of values of $d\theta/d\lambda$ until the specified initial moisture content is matched.

The second approach is directly applying Darcy equation (eq. [1]) and the law of conservation of mass (eq. [2]) in solving infiltration problems in horizontal or vertical directions. In other words, in the solution process, partial differential equations such as equations (8) and (8.1) are not needed. The basic solution strategy is dividing the soil into numerous number of layers of equal thickness. The net flux of water through each layer at any particular time is established by applying the principles of conservation of mass and Darcy's equation. The ensuing water content at each layer can thus be calculated by integrating the net flux by means of the suitable built-in functional block in SCMP (Bhuiyan, Hiler, Bavel and Aston 1971). Consequently, input data for this approach of solution are total moisture potential and conductivity curves for intended soils: For layered soils, pertinent total moisture potential and conductivity curves, for the various layers should be provided. The capability of handling soil inhomogeneity is the greatest advantage of this approach.

## APPLICATION

### Vertical Infiltration into Layered In Situ Soil

Tantalus silty clay loams are located in upper Manoa valley on the island of Oahu, Hawaii. Field hydraulic properties of this soil at various depths were investigated in situ by Ahuja and El-Swaify (1975). It was found that the Tantalus silty clay laom has homogeneous soil conditions down to the 30-cm depth or so from the surface soil. Beyond that point, inhomogeneity characteristics of the soil were observed between the various 30-cm layers down to the 6-ft depth. In considering the consistency of initial and saturated moisture content, the soil water content between adjacent layers vs. suction curves at depths of 7.6, 22.9 and 45.7cm are presented in Figures 3 and 4, and conductivity vs. water content data measurements at 30.5- and 61.0-cm depths are shown in Figures 5 and6. For demonstrative purposes, the first two 30-cm layers were considered.

Considering the first layer as an independent soil sample, the finite element and CSMP solutions were conducted (Fig. 7). For the finite element solution, the diffusivity curve was derived from the suction curves in Figure 3 by means of the following equation

$$D(\theta)=K(\theta)\frac{d\tau}{d\theta}$$

where $\tau$ is the soil-water suction (measured in cm of water depth). The resulting diffusivity curve and conductivity vs. water content data in Figure 5 are interpreted in cubic equation forms as

$$D(\theta)=-12592.7+64231.2\theta-109372.4\theta^2+62214\theta^3 \text{ cm}^2/\text{hr}$$
$$K(\theta)=-255.5+1276.9\theta-2125.1\theta^2+1177.4\theta^3 \text{ cm/hr}$$

For the CSMP solution, the suction curve for the homogeneous first layer was averaged from the drying curves in Figure 3(a) and (b), and the conductivity vs. water content data were read in pairs from Figure 5 without attempting curve-fitting.

As shown in Figure 7, the consistency of the two numerical solutions seems to verify the validity of the two different approaches.

For the two-layer soil case, which includes the top first layer and the second layer, the suction curve in Figure 4 and the conductivityvs. water content data in Figure 6 were adopted for the second layer. The resulting CSMP solutions for vertical moisture profiles for 10, 20, 40, 60, 120 and 180 minutes after water infiltration took place are plotted in Figure 8.

## DISCUSSION AND CONCLUSIONS

In atttemping to numerically solve the nonlinear moisture flow equation, the finite element method offers a convenient alternative to the finite difference method. Once the basic finite element formulation is established, the computer programming is relatively straightforward. However, in solving field problems, the efficiency of the finite element technique will hinge on the correctness of representing the diffusivity, $D(\theta)$, and conductivity, $K(\theta)$, functions of the media. For example, it was found out that equation 26 had only limited applicability, because a second degree polynomial can accurately describe the $D(\theta)$ and $K(\theta)$ functions only in the wetter portion of the $D(\theta)$ and $K(\theta)$ vs. moisture content, $\theta$, curves. Thus, equation 26 gave convergent results only when $z_i$ the initial moisture content of the soil appeared to be high. Third degree polynomial interpretation of the $D(\theta)$ and$K(\theta)$ curves enables an increase in the range of initial moisture content which would give convergent results. Some applications for Hawaiian soils show good correlation with experimental results. However, with initial moisture content less than 20%, even using third degree polynomial expressions for the $D(\theta)$ and $K(\theta)$ functions, convergent results could not be achieved. The facts seem to indicate that a better means of representation of the $D(\theta)$ and $K(\theta)$ functions of the soil media is vital to the success of the finite element technique.

Throughout the course of this study, the CSMP approach of solving the transient moisture movement in some Hawaiian soils has been demonstrated. The conciseness of the CSMP language, its being free of function-fitting in handling numerical data in the simulation processes coupled with its sophisticated internal numerical integration capability, suggest that the CSMP method (which was primarily designed for engineers and scientists working in dynamic modeling areas) might probably be the most valuable engineering approach in problem-oriented applications.

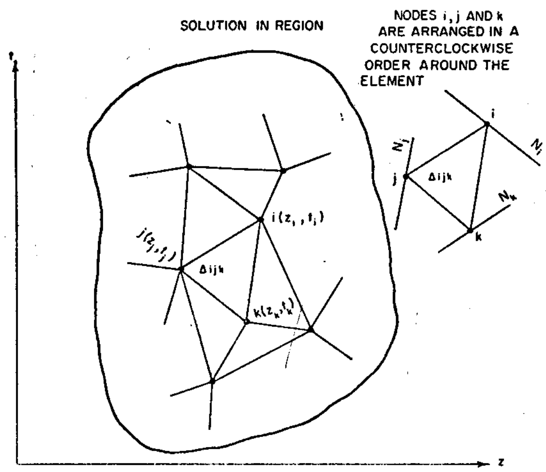
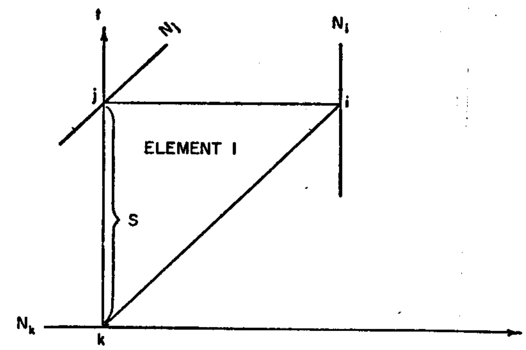
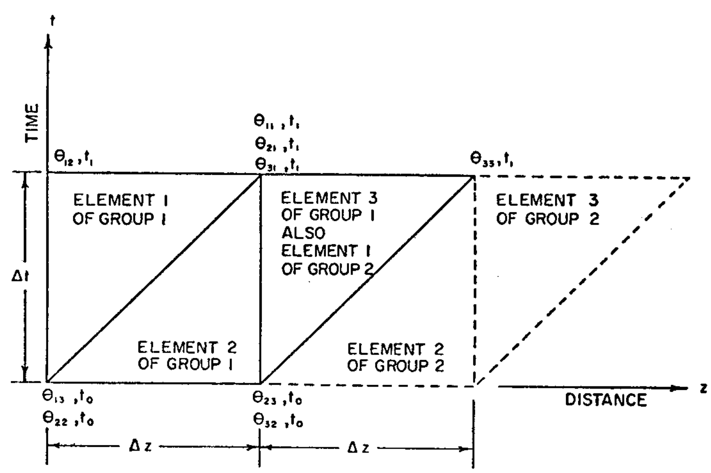



Figure 1. Division of solution region into triangular elements




a. Single triangular elements




b. Triangular elements in first time step

b. Triangular elements in first time step

Figure 2. Natural coordinate system for triangular elements

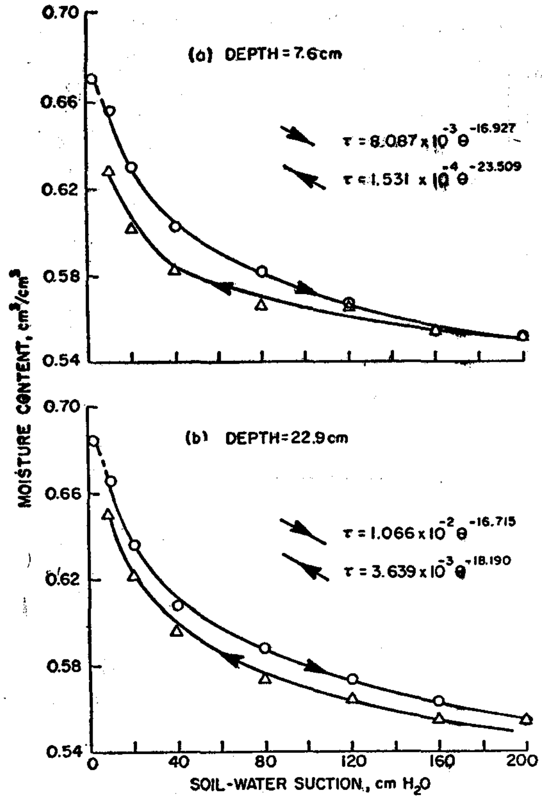



Figure 3. Soil-water content suction relationships for undisturbed soil cores taken n at 7.6- and 22.9-cm depths. The curves represent the given equations fitted to the data for wetting and drying cycles. After Ahuja and El-waifv (1975).

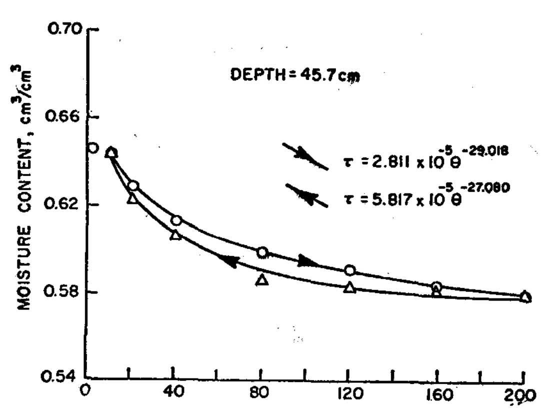



Figure 4. Soil-water content suction relationships for undisturbed soil cores taken at 45.7 and 76.2cm depths. After Ahuja and El-swaify (1975).

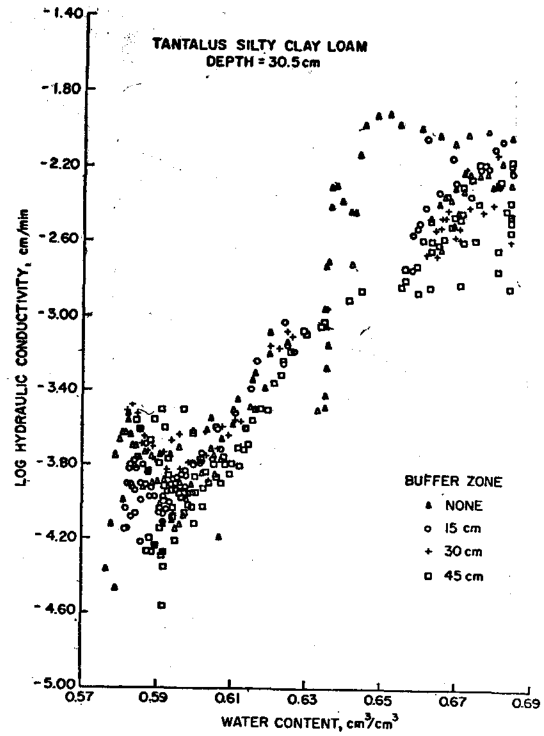



Figure 5. Hydraulic conductivity as a function of water content for 30.5-cm soil depth. The different symbols indicate dlfferent buffer zones. After Ahuja andEl-swaify (1975).

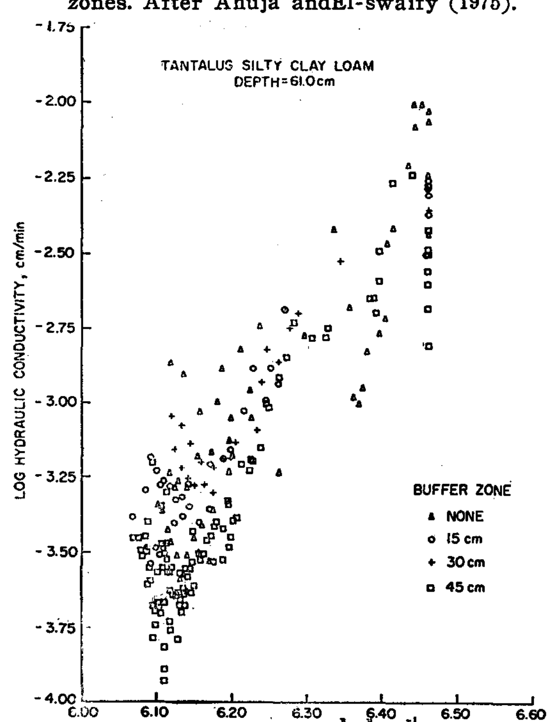



Figure 6. Hyrdraulic conductivity as a function of water content for 61.0-cm soil depth. The different symbols indicate different buffer zones. After Ahuja and El-swaify (1975).

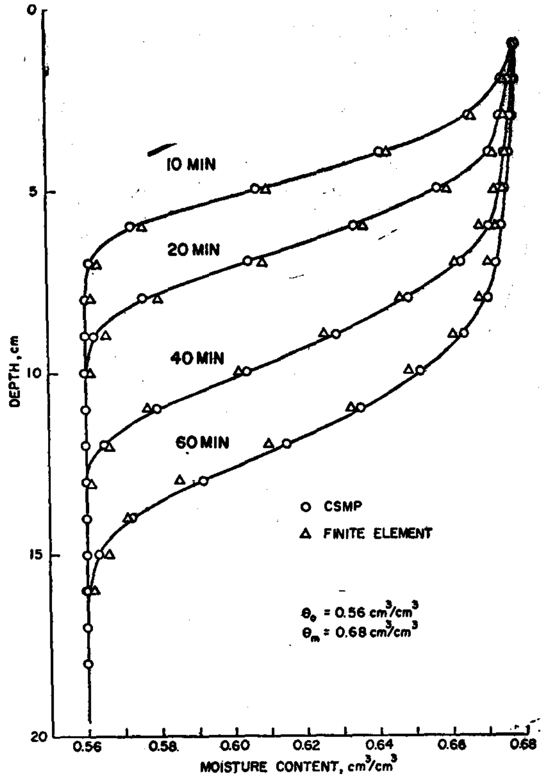



Figure 7. Calculated vertical moisture profiles for Tantalus silty clay loams

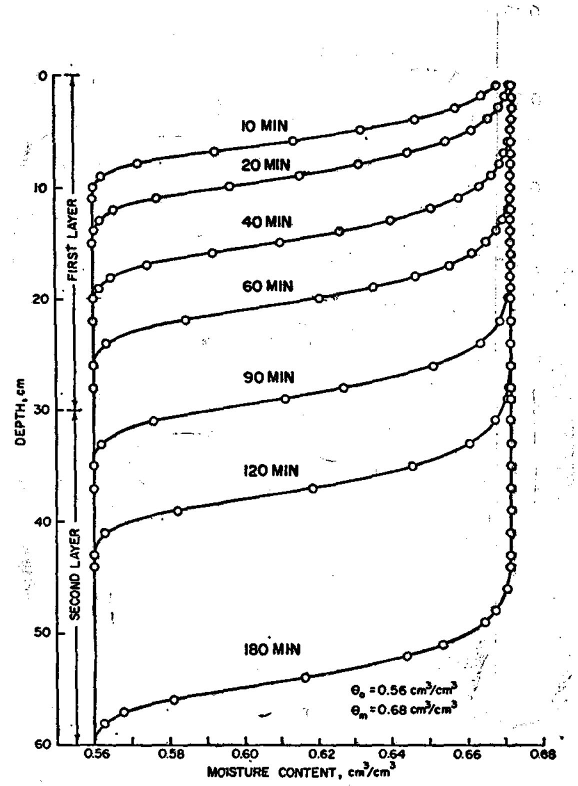



Figure 8. Calculated vertical moisture profilesfor two-layer Tantalus silty clay loams

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